Sciencia Acta Xaveriana
An International Science Journal
ISSN. 0976-1152

Volume 4
No. 1
pp. 35-70
March 2013

Domination Cover Pebbling Number for Odd Cycle Lollipop

A.Lourdusamy[1] and T.Mathivanan[2]

[1] Department of Mathematics,
St. Xavier 's College (Autonomous),
Palayamkottai - 627 002, India. E-mail: lourdugnanam@hotmail.com

[2] Research scholar,
Department of Mathematics,
St. Xavier 's College (Autonomous),
Palayamkottai-627 002, India. E-mail: tahit_van_man@yahoo.com

Abstract : Given a configuration of pebbles on the vertices of a connected graph G, a pebbling move (or pebbling step) is defined as the removal of two pebbles from a vertex and placing one pebble on an adjacent vertex. The domination cover pebbling number, $\psi(G)$, of a graph G is the minimum number of pebbles that are placed on V(G) such that after a sequence of pebbling moves, the set of vertices with pebbles forms a dominating set of G, regardless of the initial configuration. In this paper, we determine $\psi(G)$ for odd cycle lollipop.

Keywords : Pebbling, Cover pebbling, Domination cover pebbling, Lollipop.
2009 Mathematics Subject Classification : 05C99.

(Received January 2013, Accepted March 2013)

1. Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of graph pebbling [5]. There have been many developments since Hulbert's survey appeared.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from V(G) to $N \cup \{0\}$ representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected. A pebbling move (or pebbling step) is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. Define the pebbling number, π(G), to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex v, a pebble by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex v one desires to move to another root vertex, the pebbles reset to their original configuration.

The domination cover pebbling [3] is the combination of two ideas cover pebbling [2] and domination [4]. This introduces a new graph invariant called the domination cover pebbling number, ψ(G). Recall that, a set of vertices D in G is a dominating set if every vertex in G is either in D or adjacent to a vertex of D. The cover pebbling number , λ(G), is defined as the minimum number of pebbles required such that given any initial configuration of at least λ(G) pebbles, it is possible to make a series of pebbling moves to place at least one pebble on every vertex of G. The domination cover pebbling number of a graph G, proposed by A. Teguia, is the minimum number ψ(G) of pebbles required such that any initial configuration of at least ψ(G) pebbles can be transformed so that the set of vertices that contain pebbles form a dominating set of G. We have determined the domination cover pebbling number of the square of a path in [7]. In section 2, we determine the domination cover pebbling number for odd cycle lollipop. For this we use the following theorems:

Theorem 1.1[3] For n≥3, $\psi(P_n) = 2^{n+1}\left(\frac{1-8^{-(\beta_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor$, where

$n-2=\alpha_n+3\beta_n \equiv \alpha_n \pmod 3$.

■

[redacted] $- 2 = \alpha_n + 3\beta_n \equiv \alpha_n \pmod 3$.

From this theorem, we can derive the following :

$$\psi(P_n) = \begin{cases} \dfrac{2^{n+1}-1}{7}, & if\ \alpha_n = 0 \\ \dfrac{2^{n+1}-2}{7}, & if\ \alpha_n = 1 \\ \dfrac{2^{n+1}+3}{7}, & if\ \alpha_n = 2 \end{cases}$$

Also, from this we have,

$$\frac{2^{n+1}-2}{7} \le \psi(P_n) \le \frac{2^{n+1}+3}{7}.$$

Theorem 1.2[3] Let C_m be a cycle on m vertices. Then the domination cover pebbling number is given by,

$$\psi(C_m) = \begin{cases} \psi(P_k) + \psi(P_{k-1}) - |\alpha_k - 1||\alpha_{k-1} - 1|, & if\ m = 2k-2 (k \ge 3) \\ 2\psi(P_k) - |\alpha_k - 1|, & if\ m = 2k-1 (k \ge 2) \end{cases}$$

where $k-2 \equiv \alpha_k \pmod 3$ *and* $(k-1)-2 \equiv \alpha_{k-1} \pmod 3$.

2 Domination cover pebbling number for odd cycle lollipop

Definition2.1 [6] For a pair of integers m≥3 and n≥2, let L(m,n) be the lollipop graph of order n+m-1 obtained from a cycle C_m by attaching a path of length n-1 to a vertex of the cycle.

We will use the following labeling for the graphs C_m and P_n.

C_m: $v_0\, v_1\, v_2 \ldots v_{m-1}\, v_0$ (m≥3) and P_n: $v_0 v_{p_1}^{p_1} v_{p_2}^{p_2} \ldots v_{p_{n-1}}^{p_{n-1}}$ (n≥2)

If the cycle C_m in L(m,n) is odd, then L(m,n) is called odd cycle lollipop. Now, we proceed to find the domination cover pebbling number for L(3,n), where n≥2.

Theorem 2.2 Let L(3,2) be a lollipop graph. Then ψ(L(3,2))=3.

Proof: Consider the graph L(3,2). Put one pebble each on both v_1 and v_2. Clearly, we cannot cover dominate the vertex $v_{p_1}^{p_1}$. Thus, ψ(L(3,2))≥3.

Now, consider the distribution of three pebbles on the vertices of L(3,2).

Case1: C_3 contains at least one pebble.

If $v_{p_1}^{p_1}$ contains one or more pebbles then we are done, since ψ(C_3)=1. So, assume that $v_{p_1}^{p_1}$ contains zero pebbles. This implies that C_3 contains all the three pebbles. Clearly, we are done if v_0 contains a pebble. Otherwise either v_1 or v_2 contains at least two pebbles. From this we can send one pebble to v_0 and we are done.

Case2: C_3 contains zero pebbles.

This implies that $v_{p_1}^{p_1}$ contains all the three pebbles, and from this vertex we can send one pebble to v_0 and we are done.

Thus, from Case1 and Case2, ψ(L(3,2)≤3.

Therefore, ψ(L(3,2))=3. ■

■

Here after we use the following notations: consider the paths P_A: $v_0\, v_1\, v_2 \ldots v_{k-2}$ and P_B: $v_{k+1}\, v_{k+2} \ldots v_{m-1} v_0$ belonging to the cycle C_m, where m = 2k-1. Let $\hat{f}(v_i)$ be the number of pebbles at the vertex v_i and $\hat{f}(P_A)$ be the number of pebbles on the path P_A.

Consider the paths P_C: $v_{p_1} v_{p_2} ... v_{p_{n-1}}$ and P_D: $v_{p_2} v_{p_3} ... v_{p_{n-1}}$.

Theorem 2.3 Let L(3,n) be the lollipop graph, where n≥3 then,

$$\psi(L(3,n)) = \begin{cases} 2\psi(P_n)+1, if\ \alpha_n = 0\ or\ 1 \\ 2\psi(P_n)-1, if\ \alpha_n = 2 \end{cases}$$

where $n-2 \equiv \alpha_n \pmod 3$.

Proof: Consider the lollipop graph L(3,n), where n≥3 and $n-2 \equiv \alpha_n \pmod 3$.

Case1: Let α_n=0.Then n≥5.

Consider the distribution of one pebble on v_1 and $2\psi(P_n)$-1 pebbles on v_2. Clearly, we cannot cover dominate at least one of the vertices of L(3,n). Thus, $\psi(L(3,n)) \geq 2\psi(P_n)+1$.

Now, consider the distribution of $2\psi(P_n)+1$pebbles on the vertices of L(3,n).

Case1.1: C_3 contains at least one pebble.

If P_C contains $\psi(P_{n-1})$ or more pebbles then we are done, since $\psi(C_3)=1$. So, assume that P_C contains x< $\psi(P_{n-1})$ pebbles. This implies that C_3 contains at least 2 $\psi(P_n)$+1-x pebbles. Suppose we cannot move $\psi(P_n)$-x pebbles to v_0, then we must have,

$$\hat{f}(v_0) + \left\lfloor \frac{\hat{f}(v_1)}{2} \right\rfloor + \left\lfloor \frac{\hat{f}(v_2)}{2} \right\rfloor \leq [\psi(P_n) - x] - 1.$$

$$That\ is, \hat{f}(v_0) + \frac{1}{2}(\hat{f}(v_1) + \hat{f}(v_2)) \leq \psi(P_n) - x. \qquad \text{---- (1)}$$

To minimize the L.H.S of (1), it is sufficient to assume that $\hat{f}(v_0) = 0$. That is, we may assume that all the pebbles are placed at v_1 and v_2.

From (1), we get $\hat{f}(v_1)+\hat{f}(v_2)\le 2[\psi(P_n)-x]$ ---- (2)

But, we have $\hat{f}(v_1)+\hat{f}(v_2)\ge 2\psi(P_n)+1-x$ ---- (3)

The inequality in (2) contradicts the inequality in (3). So we can send $\psi(P_n)$-x pebbles to v_0 and we cover dominate the path P_n(using at most 2[$\psi(P_n)$-x] pebbles). Now C_3 contains at least, $2\psi(P_n)-x+1-[2(\psi(P_n)-x)]=x+1\ge 1$ pebbles and we are done.

Case1.2: C_3 contains zero pebbles.

This implies that P_C contains $2\psi(P_n)+1$ pebbles. We use at most 2^{n-1} pebbles to put a pebble at v_0 so that we cover dominate C_3. Since v_{p_1} is also cover dominated by v_0, we need $\psi(P_{n-2})$ pebbles in P_D. But we have enough pebbles in P_D, since $\alpha_n=0$ and

$2\psi(P_n)+1-2^{n-1}=2\left(\frac{2^{n+1}-1}{7}\right)+1-2^{n-1}=\frac{2^{n-1}+5}{7}\ge\psi(P_{n-2})$, and we are done.

Case2: Let $\alpha_n=1$. Then $n\ge 3$.

Consider the distribution of $2\psi(P_n)$ pebbles on $v_{p_{n-1}}$. Clearly, we cannot cover dominate at least one of the vertices of L(3,n). Thus, $\psi(L(3,n))\ge 2\psi(P_n)+1$.

Now, consider the distribution of $2\psi(P_n)+1$pebbles on the vertices of L(3,n).

Case2.1: C_3 contains at least one pebble.

If P_C contains $\psi(P_{n-1})$ or more pebbles then we are done, since $\psi(C_3)=1$. So, assume that P_C contains $x<\psi(P_{n-1})$ pebbles. This implies that C_3 contains at least $2\psi(P_n)+1-x$ pebbles. Suppose we cannot move $\psi(P_n)$-x pebbles to v_0, then we must have,

$$\hat{f}(v_0)+\left\lfloor\frac{\hat{f}(v_1)}{2}\right\rfloor+\left\lfloor\frac{\hat{f}(v_2)}{2}\right\rfloor\le[\psi(P_n)-x]-1.$$

$That\ is, \hat{f}(v_0) + \frac{1}{2}(\hat{f}(v_1) + \hat{f}(v_2)) \leq \psi(P_n) - x$. ---- (4)

To minimize the L.H.S of (4), it is sufficient to assume that $\hat{f}(v_0) = 0$. That is, we may assume that all the pebbles are at v_1 and v_2.

From (4), we get $\hat{f}(v_1) + \hat{f}(v_2) \leq 2[\psi(P_n) - x]$. ---- (5)

But, we have $\hat{f}(v_1) + \hat{f}(v_2) \geq 2\psi(P_n) + 1 - x$. ---- (6)

The inequality in (5) contradicts the inequality in (6). So we can send $\psi(P_n)$-x pebbles to v_0 and we cover dominate the path P_n(using at most $2[\psi(P_n)-x]$ pebbles). Now C_3 contains at least, $2\psi(P_n)-x+1-[2(\psi(P_n)-x)]=x+1\geq 1$ pebbles and we are done.

Case2.2: C_3 contains zero pebbles.

This implies that P_C contains $2\psi(P_n)+1$ pebbles. We use at most 2^{n-1} pebbles to put a pebble at v_0 so that we cover dominate C_3. Since v_{p_1} is also cover dominated by v_0, we need $\psi(P_{n-2})$ pebbles in P_D. But we have enough pebbles, since α_n=1 and

$2\psi(P_n)+1-2^{n-1} = 2\left(\frac{2^{n+1}-2}{7}\right) + 1 - 2^{n-1} \geq \frac{2^{n-1}+3}{7} \geq \psi(P_{n-2})$, and we are done.

Case3 : Let α_n=2. Then n≥4.

Consider the distribution of $2\psi(P_n)-2$ pebbles on $v_{p_{n-1}}$. Clearly, we cannot cover dominate at least one of the vertices of L(3,n). Thus,

$\psi(L(3,n)) \geq 2\psi(P_n)-1$.

Now, consider the distribution of $2\psi(P_n)-1$pebbles on the vertices of L(3,n).

Case3.1 : C_3 contains at least one pebble.

If P_C contains $\psi(P_{n-1})$ or more pebbles then we are done, since $\psi(C_3)=1$. So, assume that P_C contains $x< \psi(P_{n-1})$ pebbles. This implies that C_3 contains at least $2\psi(P_n)-1-x$ pebbles. Suppose we cannot move $\psi(P_n)-x$ pebbles to v_0, then we must have,

$$\hat{f}(v_0)+\left\lfloor \frac{\hat{f}(v_1)}{2} \right\rfloor+\left\lfloor \frac{\hat{f}(v_2)}{2} \right\rfloor \le [\psi(P_n)-x]-1.$$

That is, $\hat{f}(v_0)+\frac{1}{2}(\hat{f}(v_1)+\hat{f}(v_2)) \le \psi(P_n)-x.$ ---- (7)

To minimize the L.H.S of (7), it is sufficient to assume that $\hat{f}(v_0)=0$. That is, we may assume that all the pebbles are at v_1 and v_2.

From (7), we get $\hat{f}(v_1)+\hat{f}(v_2) \le 2[\psi(P_n)-x].$ ---- (8)

But, we have $\hat{f}(v_1)+\hat{f}(v_2) \ge 2\psi(P_n)-1-x.$ ---- (9)

The inequality in (8) contradicts the inequality in (9). So we can send $\psi(P_n)-x$ pebbles to v_0 and we cover dominate the path P_n(using at most $2[\psi(P_n)-x]$ pebbles). Now C_3 contains at least, $2\psi(P_n)-x-1-[2(\psi(P_n)-x)]=x-1\ge 1$ ($x\ge 2$) pebbles and we are done.

Case3.2: C_3 contains zero pebbles.

This implies that P_C contains $2\psi(P_n)-1$ pebbles. We use at most 2^{n-1} pebbles to put a pebble at v_0 so that we cover dominate C_3. Now we need $\psi(P_{n-2})$ pebbles in P_D. But we have enough pebbles, since $\alpha_n=2$ and $2\psi(P_n)-1-2^{n-1}=$ $2\left(\frac{2^{n+1}+3}{7}\right)-1-2^{n-1} \ge \frac{2^{n-1}-1}{7} \ge \psi(P_{n-2})$, and we are done.

Thus, from Case1, Case2 and Case3 we get,

$$\psi(L(3,n)) \le \begin{cases} 2\psi(P_n)+1, if\ \alpha_n = 0\ or\ 1 \\ 2\psi(P_n)-1, if\ \alpha_n = 2 \end{cases}$$

Therefore, $\psi(L(3,n)) = \begin{cases} 2\psi(P_n)+1, if\ \alpha_n = 0\ or\ 1 \\ 2\psi(P_n)-1, if\ \alpha_n = 2 \end{cases}$. ■

Next we proceed to find the domination cover pebbling number for L(m,2), where m=2k-1 (k≥3).

Theorem2.4 Let L(m,2) be a lollipop graph where m=2k-1(k≥3) and $k-2 \equiv \alpha_k \pmod 3$.Then $\psi(L(m,2)) = \begin{cases} 2\psi(C_m), if\ \alpha_k = 0\ or\ 2 \\ 2\psi(C_m)+1, if\ \alpha_k = 1 \end{cases}$.

Proof: Consider the lollipop graph L(m,2), where m=2k-1(k≥3) and $k-2 \equiv \alpha_k \pmod 3$.

Case1: Let α_k=0. Then k≥5.

Consider the distribution of $2\psi(C_m)$-1 pebbles on v_{p_1}, then clearly we cannot cover dominate at least one of the vertices of L(m,2). Thus, $\psi(L(m,2)) \ge 2\psi(C_m)$.

Now, consider the distribution of $2\psi(C_m)$ pebbles on the vertices of L(m,2), where α_k=0.

Case1.1: C_m contains at least $\psi(C_m)$ pebbles.

If v_{p_1} contains one or more pebbles then we are done (by our assumption). So, assume that v_{p_1} contains zero pebbles. This implies that C_m contains $2\psi(C_m)$ pebbles.

We have to send one pebble to v_0, to cover dominate the vertex v_{p_1}. Suppose we cannot send one pebble to v_0. Then we must have,

$$\hat{f}(P_A)+\left\lfloor \frac{\hat{f}(v_{k-1})+\left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-2}-1$$

and $$\hat{f}(P_B)+\left\lfloor \frac{\hat{f}(v_k)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-2}-1.$$

Adding the above inequalities, we get

$$\hat{f}(P_A)+\hat{f}(P_B)+\left\lfloor \frac{\hat{f}(v_{k-1})+\left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor+\left\lfloor \frac{\hat{f}(v_k)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-1}-2.$$

---- (10)

To minimize the L.H.S of (10), it is sufficient to assume that $\hat{f}(P_A)$=0= $\hat{f}(P_B)$. That is, we may assume that all the pebbles are at $v_{k-1}$and v_k.Now, 2ψ(C_m) is even, so both $\hat{f}(v_{k-1})$ *and* $\hat{f}(v_k)$ are either odd or even.

Subcase1 (a): Suppose, both $\hat{f}(v_{k-1})$ *and* $\hat{f}(v_k)$ are odd.

From (10), we get $$\left\lfloor \frac{\hat{f}(v_{k-1})+\left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor+\left\lfloor \frac{\hat{f}(v_k)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-1}-2.$$

That is, $\dfrac{(\hat{f}(v_{k-1})-1)+\left(\dfrac{\hat{f}(v_k)-1}{2}\right)}{2}+\dfrac{(\hat{f}(v_k)-1)+\left(\dfrac{\hat{f}(v_{k-1})-1}{2}\right)}{2}\leq 2^{k-1}-2$.

That is, $\dfrac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)-2\right]\leq 2^{k-1}-2$. ---- (11)

But, we have, $\hat{f}(v_{k-1})+\hat{f}(v_k)\geq 2\psi(C_m)$

$$=2\left[2\psi(P_k)-|\alpha_k-1|\right],\ since\ m=2k-1$$

$$=4\left(\frac{2^{k+1}-1}{7}\right)-2,\ since\ \alpha_k=0$$

$$=\frac{4(2^{k+1})-18}{7}.$$

That is, $\hat{f}(v_{k-1})+\hat{f}(v_k)\geq\dfrac{8(2^k-1)-10}{7}$.

Thus,

$$\frac{3}{4}\left(\hat{f}(v_{k-1})+\hat{f}(v_k)-2\right)\geq\frac{3}{4}\left(\frac{8(2^k-1)-10}{7}-2\right)=6\left(\frac{2^k-4}{7}\right)$$

$$\geq 2^{k-1}\text{ , since k}\geq 5. \qquad \text{---- (12)}$$

The inequality in (11) contradicts the inequality in (12).

Subcase1 (b): Suppose, both $\hat{f}(v_{k-1})\ and\ \hat{f}(v_k)$ are even.

From (10), we get $\left\lfloor\dfrac{\hat{f}(v_{k-1})+\left\lfloor\dfrac{\hat{f}(v_k)}{2}\right\rfloor}{2}\right\rfloor+\left\lfloor\dfrac{\hat{f}(v_k)+\left\lfloor\dfrac{\hat{f}(v_{k-1})}{2}\right\rfloor}{2}\right\rfloor\leq 2^{k-1}-2$.

That is, $\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)\right]\leq 2^{k-1}-1$. ---- (13)

But, we have, $\hat{f}(v_{k-1})+\hat{f}(v_k)\geq 2\psi(C_m)$ and since m=2k-1 and α_k=0, we get

$$\frac{3}{4}\left(\hat{f}(v_{k-1})+\hat{f}(v_k)\right)\geq\frac{3}{4}\left(\frac{8(2^k-1)-10}{7}\right)$$

$$\geq 2^{k-1}\text{, since } k\geq 5. \qquad \text{---- (14)}$$

The inequality in (13) contradicts the inequality in (14).

From the Subcase1 (a) and Subcase1 (b) , we can send one pebble to v_0 using at most 2^{k-1} pebbles.

Now, the minimum number of pebbles that C_m contains is

$$2\psi(C_m)-2^{k-1}=\psi(C_m)+\left[2\left(\frac{2^{k+1}-1}{7}\right)-1\right]-2^{k-1}$$

$$=\psi(C_m)+\left(\frac{2^{k-1}-9}{7}\right)$$

$$\geq\psi(C_m),$$

where the first equality follows since m=2k-1 and α_k=0 and the third inequality follows since k≥5. Thus, we have enough pebbles to cover dominate C_m and we are done.

Case1.2 : C_m contains x<ψ(C_m) pebbles.

This implies that, v_{p_1} contains at least 2ψ(C_m) –x pebbles. We can send ψ(C_m)-$\left\lfloor\frac{x}{2}\right\rfloor$ pebbles to v_0. So, C_m contains at least x+ ψ(C_m)- $\left\lfloor\frac{x}{2}\right\rfloor$ ≥ψ(C_m) pebbles and we are done.

Case2 : Let α_k=2. Then k≥4.

Consider the distribution of $2\psi(C_m)-1$ pebbles on v_{p_1}, then clearly we cannot cover dominate at least one of the vertices of L(m,2). Thus, $\psi(L(m,2)) \geq 2\psi(C_m)$.

Now, consider the distribution of $2\psi(C_m)$ pebbles on the vertices of L(m,2), where α_k=2.

Case2.1 : C_m contains at least $\psi(C_m)$ pebbles.

If v_{p_1} contains one or more pebbles then we are done (by our assumption). So, assume that v_{p_1} contains zero pebbles. This implies that C_m contains $2\psi(C_m)$ pebbles. We have to send one pebble to v_0, to cover dominate the vertex v_{p_1}. Suppose we cannot send one pebble to v_0. Then we must have,

$$\hat{f}(P_A)+\hat{f}(P_B)+\left\lfloor \frac{\hat{f}(v_{k-1})+\left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor + \left\lfloor \frac{\hat{f}(v_k)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \leq 2^{k-1}-2 .$$

---- (15)

To minimize the L.H.S of (15), it is sufficient to assume that $\hat{f}(P_A)=0=\hat{f}(P_B)$. That is, we may assume that all the pebbles are at v_{k-1} and v_k. Now, $2\psi(C_m)$ is even, so both $\hat{f}(v_{k-1})$ *and* $\hat{f}(v_k)$ are either odd or even.

Subcase2 (a) : Suppose, both $\hat{f}(v_{k-1})$ *and* $\hat{f}(v_k)$ are odd.

From (15), we get $\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)-2\right] \leq 2^{k-1}-2.$ ---- (16)

But, we have $\hat{f}(v_{k-1})+\hat{f}(v_k) \geq 2\psi(C_m) = \frac{4(2^{k+1})-2}{7}.$

But, we have $\hat{f}(v_{k-1})+\hat{f}(v_k)\geq 2\psi(C_m)=\dfrac{4(2^{k+1})-2}{7}$.

That is, $\hat{f}(v_{k-1})+\hat{f}(v_k)\geq 2\left(\dfrac{4(2^k)-1}{7}\right)$.

Thus, $\dfrac{3}{4}\left(\hat{f}(v_{k-1})+\hat{f}(v_k)-2\right)\geq\dfrac{3}{4}\left[2\left(\dfrac{4(2^k)-1}{7}\right)-2\right]$

$$\geq 2^{k-1}\text{, since } k\geq 4. \qquad \text{---- (17)}$$

The inequality in (16) contradicts the inequality in (17).

Subcase2 (b): Suppose, both $\hat{f}(v_{k-1})$ *and* $\hat{f}(v_k)$ are even.

From (15), we get

$$\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)\right]\leq 2^{k-1}-1. \qquad \text{---- (18)}$$

But, we have, $\hat{f}(v_{k-1})+\hat{f}(v_k)\geq 2\psi(C_m)$ and since m=2k-1 and α_k=2, we get

$$\frac{3}{4}\left(\hat{f}(v_{k-1})+\hat{f}(v_k)\right)\geq\frac{3}{4}\left[2\left(\frac{4(2^k)-1}{7}\right)\right]$$

$$\geq 2^{k-1}\text{, since } k\geq 4. \qquad \text{---- (19)}$$

The inequality in (18) contradicts the inequality in (19).

From the Subcase2 (a) and Subcase2 (b), we can send one pebble to v_0 using at most 2^{k-1} pebbles.

Now, the minimum number of pebbles that C_m contains is

$2\psi(C_m)-2^{k-1} = \psi(C_m) + \left[2\left(\frac{2^{k+1}+3}{7}\right)-1\right] - 2^{k-1}, \textit{since } m = 2k-1 \textit{ and } \alpha_k = 2$

$\geq \psi(C_m), \textit{since } k \geq 4.$

Thus, we have enough pebbles to cover dominate C_m and we are done.

Case2.2 : C_m contains $x<\psi(C_m)$ pebbles.

This implies that, v_{p_1} contains at least $2\psi(C_m) - x$ pebbles. We can send $\psi(C_m)-\left\lfloor \frac{x}{2} \right\rfloor$ pebbles to v_0. So, C_m contains at least $x+\psi(C_m)-\left\lfloor \frac{x}{2} \right\rfloor \geq \psi(C_m)$ pebbles and we are done.

Case3: Let $\alpha_k=1$. Then $k\geq 3$.

Consider the distribution of $2\psi(C_m)$ pebbles on v_{p_1}. Then clearly we cannot cover dominate at least one of the vertices of L(m,2).

Thus, $\psi(L(m,2)) \geq 2\psi(C_m)+1$.

Now, consider the distribution of $2\psi(C_m)+1$ pebbles on the vertices of L(m,2), where $\alpha_k=1$.

Case3.1: C_m contains at least $\psi(C_m)$ pebbles.

If v_{p_1} contains one or more pebbles then we are done (by our assumption). So, assume that v_{p_1} contains zero pebbles. This implies that C_m contains $2\psi(C_m)+1$ pebbles. We have to send one pebble to v_0, to cover dominate the vertex v_{p_1}. Suppose we cannot send one pebble to v_0. Then we must have,

$$\hat{f}(P_A)+\hat{f}(P_B)+\left\lfloor \frac{\hat{f}(v_{k-1})+\left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor+\left\lfloor \frac{\hat{f}(v_k)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-1}-2.$$

---- (20)

To minimize the L.H.S of (20), it is sufficient to assume that $\hat{f}(P_A)$=0= $\hat{f}(P_B)$. That is, we may assume that all the pebbles are at $v_{k-1}$and v_k. Now, $2\psi(C_m)+1$ is odd, so exactly one of $\hat{f}(v_{k-1})$, $\hat{f}(v_k)$ is even. Without loss of generality assume $\hat{f}(v_{k-1})$ is even.

From (20), we get

$$\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)\right]-\frac{5}{4} \le 2^{k-1}-2. \quad \text{---- (21)}$$

But, we have, $\hat{f}(v_{k-1})+\hat{f}(v_k) \ge 2\psi(C_m)+1$. Then,

$$\frac{3}{4}\left(\hat{f}(v_{k-1})+\hat{f}(v_k)\right)-\frac{5}{4} \ge 2^{k-1} \quad \text{---- (22)}$$

The inequality in (21) contradicts the inequality in (22). So, we can send one pebble to v_0 using at most 2^{k-1} pebbles.

Now, the minimum number of pebbles that C_m contains is

$2\psi(C_m)+1-2^{k-1} \ge \psi(C_m)$.

Thus, we have enough pebbles to cover dominate C_m and we are done.

Case3.2: C_m contains $x<\psi(C_m)$ pebbles.

This implies that, v_{p_1} contains at least 2ψ(C_m) –x pebbles. We can send ψ(C_m)- $\left\lfloor \frac{x}{2} \right\rfloor$ pebbles to v_0. So, C_m contains at least x+ ψ(C_m)- $\left\lfloor \frac{x}{2} \right\rfloor$ ≥ψ(C_m) pebbles and we are done.

Thus, $\psi(L(m,2)) \le \begin{cases} 2\psi(C_m), if\ \alpha_k = 0\ or\ 2 \\ 2\psi(C_m)+1, if\ \alpha_k = 1 \end{cases}$.

Therefore, $\psi(L(m,2)) = \begin{cases} 2\psi(C_m), if\ \alpha_k = 0\ or\ 2 \\ 2\psi(C_m)+1, if\ \alpha_k = 1 \end{cases}$.

where m=2k-1(k≥3) and k-2 ≡ α_k(mod 3). ■

Next, we proceed to find the domination cover pebbling number of L(m,n), where m=2k-1 (k≥3) and n≥3.

Theorem2.5 Let L(m,n) be a lollipop graph where m=2k-1 (k≥ 3) and n≥3. Then,

$$\psi(L(m,n)) = \begin{cases} 2^{n-1}\psi(C_m)+\psi(P_{n-1}), if\ \alpha_k = 1 \\ 2^{n-1}\psi(C_m)+\psi(P_{n-2}), if\ \alpha_k = 0\ or 2 \end{cases}$$

where k-2 ≡ α_k(mod 3).

Proof: Consider the lollipop graph L(m,n), where m=2k-1(k≥3) and n≥3.

Case1: Let α_k=1. Then k≥3.

Consider the distribution of ψ(L(m,n))-1pebbles at $v_{p_{n-1}}$. Clearly, we cannot cover dominate at least one of the vertices of L(m,n).

Thus, $\psi(L(m,n)) \ge 2^{n-1}\psi(C_m)+\psi(P_{n-1})$.

Now, consider the distribution of $\psi(L(m,n))$ pebbles on the vertices of L(m,n).

Case1.1: C_m contains at least $\psi(C_m)$ pebbles.

If P_C contains $\psi(P_{n-1})$ pebbles are more, then clearly we are done(by our assumption). So assume that P_C contains $x< \psi(P_{n-1})$ pebbles. This implies that, C_m contains $2^{n-1}\psi(C_m)+\psi(P_{n-1})-x$ pebbles. Suppose, we cannot move $\psi(P_n)$-x pebbles to v_0, then we must have,

$$\hat{f}(P_A)+\left\lfloor \frac{\hat{f}(v_{k-1})+\left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-2}\left[\psi(P_n)-x\right]-1$$

$$and\ \hat{f}(P_B)+\left\lfloor \frac{\hat{f}(v_k)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-2}\left[\psi(P_n)-x\right]-1.$$

Adding the above inequalities, we get

$$\hat{f}(P_A)+\hat{f}(P_B)+\left\lfloor \frac{\hat{f}(v_{k-1})+\left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor + \left\lfloor \frac{\hat{f}(v_k)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-1}\left[\psi(P_n)-x\right]-2. \quad \text{---- (23)}$$

To minimize the L.H.S of (23), it is sufficient to assume that $\hat{f}(P_A)$=0= $\hat{f}(P_B)$. That is, we may assume that all the pebbles are at v_k and v_{k-1}.

Now, $2^{n-1}\psi(C_m)+\psi(P_{n-1})-x$ is odd or even, since it depends on both

$\psi(P_{n-1})$ and x.

Subcase1 (a): Suppose, $\hat{f}(v_{k-1})+\hat{f}(v_k)$ is even.

This implies that, both $\hat{f}(v_{k-1})$ *and* $\hat{f}(v_k)$ are odd or even. Suppose, both $\hat{f}(v_{k-1})$ *and* $\hat{f}(v_k)$ are odd, then from (23), we get

$$\left\lfloor \frac{\hat{f}(v_{k-1})+\left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor + \left\lfloor \frac{\hat{f}(v_k)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-1}\left[\psi(P_n)-x\right]-2.$$

That is, $\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)-2\right]\le 2^{k-1}\left[\psi(P_n)-x\right]-2.$ ---- (24)

But, we have $\hat{f}(v_k)+\hat{f}(v_{k-1})\ge 2^{n-1}\psi(C_m)+\psi(P_{n-1})-x$

$=2^{n-1}\left[2\psi(P_k)-|\alpha_k-1|\right]+\psi(P_{n-1})-x$

$\ge 2^n\left[\frac{2^{k+1}-2}{7}\right]+\frac{2^n-2}{7}-x$, where the second equality follows since m=2k-1,

and the third inequality follows since $\alpha_k=1$ *and* $\psi(P_n)\ge\frac{2^{n+1}-2}{7}$.

Then $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})-2\right]\ge\frac{3}{4}\left[2^n\left[\frac{2^{k+1}-2}{7}\right]+\frac{2^n-2}{7}\right]-\frac{3}{4}x-\frac{3}{2}$

$$= \frac{3}{4}\left[\frac{2^k(2^{n+1}+3)}{7} - \frac{3(2^k)}{7} - \frac{2^n}{7}\right] - \frac{3}{4}x - \frac{24}{14}$$

$$\geq 3(2^{k-2})\psi(P_n) - \frac{3}{4}\left[\frac{3(2^k)}{7} + \frac{2^n}{7}\right] - \frac{3}{4}x - \frac{24}{14}$$

$$\geq 2^{k-1}\psi(P_n) + 2^{k-2}\left[\frac{2^{n+1}-2}{7} - \frac{9(2^k)}{28(2^{k-2})} - \frac{3(2^n)}{28(2^{k-2})}\right] - \frac{3}{4}x - \frac{12}{7}$$

$$\geq 2^{k-1}\psi(P_n) + 2^{k-2}\left[\frac{2^{n+1}-2}{7} - \frac{36}{28(2)} - \frac{3(2^n)}{28(2)}\right] - \frac{3}{4}x - \frac{12}{7}$$

$$= 2^{k-1}\psi(P_n) + 2^{k-2}\left[\frac{13(2^n)-52}{56}\right] - x - 2,$$

where the third inequality follows since $\psi(P_n) \leq \frac{2^{n+1}+3}{7}$.

That is, $\frac{3}{4}\left[\hat{f}(v_k) + \hat{f}(v_{k-1}) - 2\right] \geq 2^{k-1}\psi(P_n) - x - 2$. ---- (25)

The inequality in (24) contradicts the inequality in (25).

Suppose, both $\hat{f}(v_{k-1})$ *and* $\hat{f}(v_k)$ are even, then from (23), we get

$$\frac{(\hat{f}(v_{k-1})-1) + \left(\frac{\hat{f}(v_k)}{2}\right)}{2} + \frac{(\hat{f}(v_k)-1) + \left(\frac{\hat{f}(v_{k-1})}{2}\right)}{2} \leq 2^{k-1}\left[\psi(P_n) - x\right] - 2.$$

That is, $\frac{3}{4}\left[\hat{f}(v_{k-1}) + \hat{f}(v_k)\right] \leq 2^{k-1}\left[\psi(P_n) - x\right] - 1.$ ---- (26)

But, we have $\hat{f}(v_k)+\hat{f}(v_{k-1}) \geq 2^{n-1}\psi(C_m)+\psi(P_{n-1})-x$

$=2^{n-1}\left[2\psi(P_k)-|\alpha_k-1|\right]+\psi(P_{n-1})-x$

$\geq 2^n\left[\frac{2^{k+1}-2}{7}\right]+\frac{2^n-2}{7}-x$, where the second equality follows since m=2k-1,

and the third inequality follows since $\alpha_k=1 \; and \; \psi(P_n)\geq \frac{2^{n+1}-2}{7}$.

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right]\geq \frac{3}{4}\left[2^n\left[\frac{2^{k+1}-2}{7}\right]+\frac{2^n-2}{7}\right]-\frac{3}{4}x$

$\geq 2^{k-1}\psi(P_n)+2^{k-2}\left[\frac{13(2^n)-52}{56}\right]-x-2.$

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right]\geq 2^{k-1}\psi(P_n)-x-1$. ---- (27)

The inequality in (26) contradicts the inequality in (27).

Subcase1(b): If $\hat{f}(v_{k-1})+\hat{f}(v_k)$ is odd.

Without loss of generality, let $\hat{f}(v_k)$ be odd. Then $\hat{f}(v_{k-1})$ is even.

From (23), we get

$$\frac{(\hat{f}(v_{k-1})-1)+\left(\frac{\hat{f}(v_k)-1}{2}\right)}{2}+\frac{(\hat{f}(v_k)-1)+\left(\frac{\hat{f}(v_{k-1})}{2}\right)}{2}\leq 2^{k-1}\left[\psi(P_n)-x\right]-2.$$

That is, $\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)\right]-\frac{5}{4}\leq 2^{k-1}\left[\psi(P_n)-x\right]-2$. ---- (28)

But, we have $\hat{f}(v_k)+\hat{f}(v_{k-1})\geq 2^{n-1}\psi(C_m)+\psi(P_{n-1})-x$

$=2^{n-1}\left[2\psi(P_k)-|\alpha_k-1|\right]+\psi(P_{n-1})-x$

$\geq 2^n\left[\frac{2^{k+1}-2}{7}\right]+\frac{2^n-2}{7}-x$, where the second equality follows since m=2k-1, and the third inequality follows since $\alpha_k=1\, and\, \psi(P_n)\geq\frac{2^{n+1}-2}{7}$.

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right]-\frac{5}{4}\geq\frac{3}{4}\left[2^n\left(\frac{2^{k+1}-2}{7}\right)+\frac{2^n-2}{7}\right]-\frac{3}{4}x-\frac{5}{4}$

$\geq 2^{k-1}\psi(P_n)+2^{k-2}\left[\frac{13(2^n)-52}{56}\right]-x-2$.

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right]-\frac{5}{4}\geq\; 2^{k-1}\psi(P_n)-x-2$. ---- (29)

The inequality in (28) contradicts the inequality in (29).

From Subcase 1(a) and Subcase 1(b), we can always send $\psi(P_n)-x$ pebbles to v_0 at a cost of at most $2^{k-1}[\psi(P_n)-x]$ pebbles. Thus, we cover dominate the path P_n. Now, we have to cover dominate C_m. In C_m, we have at least $2^{n-1}\psi(C_m)+\psi(P_{n-1})-x-2^{k-1}\left[\psi(P_n)-x\right]$ pebbles. We need at most $\psi(C_m)$ pebbles to cover dominate C_m. But,

$2^{n-1}\psi(C_m)+\psi(P_{n-1})-x-2^{k-1}\left[\psi(P_n)-x\right]-\psi(C_m)$

$=(2^{n-1}-1)\psi(C_m)+\psi(P_{n-1})-x-2^{k-1}\left[\psi(P_n)-x\right]$

$=(2^{n-1}-1)\left[2\psi(P_k)\right]-2^{k-1}\psi(P_n)+\psi(P_{n-1})+(2^{k-1}-1)x$

$$\geq (2^{n-1}-1)\left(\frac{2(2^{k+1}-2)}{7}\right)-2^{k-1}\left(\frac{2^{n+1}+3}{7}\right)$$

$$= 2^{k-1}\left[\frac{4(2^n)-8-2(2^n)-3}{7}-\frac{4(2^n)-4}{7(2^{k-1})}\right]$$

$$\geq 2^{k-1}\left[\frac{8(2^n)-4(2^n)-15}{28}\right]$$

$$\geq 2^{k-1}\left[\frac{4(2^n)-15}{28}\right]>0,$$

where the second equality follows since m=2k-1, the third inequality follows since $\alpha_k = 1 \; and \; \psi(P_n) \leq \frac{2^{n+1}+3}{7}$, the fifth inequality follows since k≥3, and the sixth inequality follows $since \; n > 2 \; and \; k > 2$.

Thus, we have enough pebbles to cover dominate C_m and hence we are done.

Case1.2: C_m contains $y<\psi(C_m)$ pebbles.

This implies that, P_C contains $2^{n-1}\psi(C_m)+\psi(P_{n-1})-y$ pebbles. We use at most $\psi(P_{n-1})$ pebbles to cover dominate P_C. Thus, we have at least $2^{n-1}\psi(C_m)-y$ pebbles in P_C. We need at most 2^{n-1} $[\psi(C_m)-y]$ pebbles from P_C to cover the vertices of C_m. But,

$$2^{n-1}\psi(C_m)-y-2^{n-1}[\psi(C_m)-y]$$

$$=(2^{n-1}-1)y>0,$$

where the second inequality follows since n>2. Thus, we can send $\psi(C_m)$-y pebbles to v_0 and already C_m contains y pebbles implies that C_m contains $\psi(C_m)$ pebbles and we are done.

So, $\psi\,(L(m,n)) \le 2^{n-1}\psi\,(C_m) + \psi\,(P_{n-1})$.

Therefore, $\psi\,(L(m,n)) = 2^{n-1}\psi\,(C_m) + \psi\,(P_{n-1})$, if α_k=1.

Case2: Let α_k=2. Then k≥4.

Consider the distribution of ψ(L(m,n))-1pebbles at $v_{p_{n-1}}$. Clearly, we cannot cover dominate at least one of the vertices of L(m,n).

Thus, $\psi\,(L(m,n)) \ge 2^{n-1}\psi\,(C_m) + \psi\,(P_{n-2})$.

Now, consider the distribution of ψ(L(m,n)) pebbles on the vertices of L(m,n).

Case2.1: C_m contains at least $\psi(C_m)$ pebbles.

If P_C contains $\psi(P_{n-1})$ pebbles are more, then clearly we are done(by our assumption). So assume that P_C contains x< $\psi(P_{n-1})$ pebbles. This implies that, C_m contains $2^{n-1}\psi\,(C_m) + \psi\,(P_{n-2}) - x$ pebbles. Suppose, we cannot move $\psi(P_n)$-x pebbles to v_0, then we must have,

$$\hat{f}(P_A) + \hat{f}(P_B) + \left\lfloor \frac{\hat{f}(v_{k-1}) + \left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor$$

$$+ \left\lfloor \frac{\hat{f}(v_k) + \left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \le 2^{k-1}\left[\psi\,(P_n) - x\right] - 2. \quad \text{---- (30)}$$

To minimize the L.H.S of (30), it is sufficient to assume that $\hat{f}(P_A)$=0= $\hat{f}(P_B)$. That is, we may assume that all the pebbles are at v_k and v_{k-1}.

Now, $2^{n-1}\psi(C_m)+\psi(P_{n-2})-x$ is odd or even, since it depends on both $\psi(P_{n-2})$ and x.

Subcase2 (a): Suppose, $\hat{f}(v_{k-1})+\hat{f}(v_k)$ is even.

This implies that, both $\hat{f}(v_{k-1})\ and\ \hat{f}(v_k)$ are odd or even. Suppose, both $\hat{f}(v_{k-1})\ and\ \hat{f}(v_k)$ are odd, then

From (30), we get

$$\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)-2\right]\le 2^{k-1}\left[\psi(P_n)-x\right]-2. \text{ ---- (31)}$$

But, we have $\hat{f}(v_k)+\hat{f}(v_{k-1})\ge 2^{n-1}\psi(C_m)+\psi(P_{n-2})-x$

$$= 2^{n-1}\left[2\psi(P_k)-|\alpha_k-1|\right]+\psi(P_{n-2})-x$$

$$\ge 2^n\left[\frac{2^{k+1}+3}{7}-\frac{1}{2}\right]+\frac{2^{n-1}-2}{7}-x,$$

where the second equality follows since m=2k-1, and the third inequality follows since $\alpha_k=2\ and\ \psi(P_n)\ge\frac{2^{n+1}-2}{7}$.

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})-2\right]\ge\frac{3}{4}\left[2^n\left[\frac{2^{k+1}+3}{7}-\frac{1}{2}\right]+\frac{2^{n-1}-2}{7}\right]-\frac{3}{4}x-\frac{3}{2}$

$$\ge 2^{k-1}\psi(P_n)+2^{k-2}\psi(P_n)-\frac{3}{4}\left[\frac{3(2^k)}{7}\right]-\frac{3}{4}x-2$$

$$\geq 2^{k-1}\psi(P_n) + 2^{k-2}\left[\frac{2^{n+1}-2}{7} - \frac{3}{4}\left(\frac{12}{7}\right)\right] - \frac{3}{4}x - 2$$

$$\geq 2^{k-1}\psi(P_n) - x - 2,$$

where the second inequality follows $since \psi(P_n) \leq \frac{2^{n+1}+3}{7}$, and the fourth inequality follows $since\ n \geq 3\ and\ k \geq 4$.

That is, $\frac{3}{4}\left[\hat{f}(v_k) + \hat{f}(v_{k-1}) - 2\right] \geq 2^{k-1}\psi(P_n) - x - 2$. ---- (32)

The inequality in (31) contradicts the inequality in (32).

Suppose, both $\hat{f}(v_{k-1})\ and\ \hat{f}(v_k)$ are even, then

From (30), we get

$$\frac{3}{4}\left[\hat{f}(v_{k-1}) + \hat{f}(v_k)\right] \leq 2^{k-1}\left[\psi(P_n) - x\right] - 1. \quad \text{---- (33)}$$

But, we have $\hat{f}(v_k) + \hat{f}(v_{k-1}) \geq 2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x$

$$\geq 2^n\left[\frac{2^{k+1}+3}{7} - \frac{1}{2}\right] + \frac{2^{n-1}-2}{7} - x,$$

where the second inequality follows since $\alpha_k = 2\ and\ \psi(P_n) \geq \frac{2^{n+1}-2}{7}$.

That is, $\frac{3}{4}\left[\hat{f}(v_k) + \hat{f}(v_{k-1})\right] \geq \frac{3}{4}\left[2^n\left[\frac{2^{k+1}+3}{7} - \frac{1}{2}\right] + \frac{2^{n-1}-2}{7}\right] - \frac{3}{4}x$

$$\geq 2^{k-1}\psi(P_n)+2^{k-2}\psi(P_n)-\frac{3}{4}\left[\frac{3(2^k)}{7}\right]-x-1$$

$$\geq 2^{k-1}\psi(P_n)+2^{k-2}\left[\frac{2^{n+1}-2}{7}-\frac{3}{4}\left(\frac{12}{7}\right)\right]-x-1$$

$$\geq 2^{k-1}\psi(P_n)-x-1,$$

where the second inequality follows since $\psi(P_n)\leq\frac{2^{n+1}+3}{7}$, and the fourth inequality follows $since\; n\geq 3\, and\; k\geq 4.$

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right]\geq 2^{k-1}\psi(P_n)-x-1$. ---- (34)

The inequality in (33) contradicts the inequality in (34).

Subcase2 (b): If $\hat{f}(v_{k-1})+\hat{f}(v_k)$ is odd.

Without loss of generality, let $\hat{f}(v_k)$ be odd. Then $\hat{f}(v_{k-1})$ is even.

From (30), we get

$$\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)\right]-\frac{5}{4}\leq 2^{k-1}\left[\psi(P_n)-x\right]-2. \qquad \text{---- (35)}$$

But, we have $\hat{f}(v_k)+\hat{f}(v_{k-1})\geq 2^{n-1}\psi(C_m)+\psi(P_{n-2})-x$

$$\geq 2^n\left[\frac{2^{k+1}+3}{7}-\frac{1}{2}\right]+\frac{2^{n-1}-2}{7}-x,$$

where the second equality follows since m=2k-1, and the third inequality follows since $\alpha_k = 2 \; and \; \psi(P_n) \geq \frac{2^{n+1}-2}{7}$.

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right]-\frac{5}{4} \geq \frac{3}{4}\left[2^n\left(\frac{2^{k+1}+3}{7}-\frac{1}{2}\right)+\frac{2^{n-1}-2}{7}\right]-\frac{3}{4}x-\frac{5}{4}$

$= 2^{k-1}\psi(P_n)+2^{k-2}\left[\frac{2^{n+1}-11}{7}\right]-x-2$.

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right]-\frac{5}{4} \geq 2^{k-1}\psi(P_n)-x-2$. ---- (36)

The inequality in (35) contradicts the inequality in (36).

From Subcase2 (a) and Subcase2 (b), we can always send $\psi(P_n)-x$ pebbles to v_0 at a cost of at most $2^{k-1}[\psi(P_n)-x]$ pebbles. Thus, we cover dominate the path P_n. Now, we have to cover dominate C_m. In C_m, we have at least $2^{n-1}\psi(C_m)+\psi(P_{n-2})-x-2^{k-1}\left[\psi(P_n)-x\right]$ pebbles. We need at most $\psi(C_m)$ pebbles to cover dominate C_m. But,

$$2^{n-1}\psi(C_m)+\psi(P_{n-2})-x-2^{k-1}\left[\psi(P_n)-x\right]-\psi(C_m)$$

$$= (2^{n-1}-1)\left[2\psi(P_k)-1\right]-2^{k-1}\psi(P_n)+\psi(P_{n-2})+(2^{k-1}-1)x$$

$$\geq (2^{n-1}-1)\left(\frac{2(2^{k+1}+3)}{7}-1\right)-2^{k-1}\left(\frac{2^{n+1}+3}{7}\right)$$

$$= 2^{k-1}\left[\frac{4(2^n)-8-2(2^n)-3}{7}-\frac{2^{n-1}-1}{7(2^{k-1})}\right]$$

$$\geq 2^{k-1}\left[\frac{31(2^{n-1})-87}{56}\right] > 0,$$

where the second equality follows since $\alpha_k = 1 \; and \; \psi(P_n) \leq \frac{2^{n+1}+3}{7}$, and the fourth inequality follows $since \; n > 2 \; and \; k > 3$.

Thus, we have enough pebbles to cover dominate C_m and hence we are done.

Case2.2: C_m contains $y<\psi(C_m)$ pebbles.

This implies that, P_C contains $2^{n-1}\psi(C_m)+\psi(P_{n-2})-y$ pebbles. We use at most $\psi(P_{n-1})$ pebbles to cover dominate P_C. Thus, we have at least $2^{n-1}\psi(C_m)+\psi(P_{n-2})-y-\psi(P_{n-1})$ pebbles in P_C. We need at most $2^{n-1}[\psi(C_m)-y]$ pebbles from P_C to cover dominate the vertices of C_m. But,

$$2^{n-1}\psi(C_m)+\psi(P_{n-2})-y-\psi(P_{n-1})-2^{n-1}\left[\psi(C_m)-y\right]$$

$$\geq 2^{n-1}y+\frac{2^{n-1}-2}{7}-y-\frac{2^n+3}{7}$$

$$\geq 2^{n-1}\left[y-\frac{5+7y}{7(4)}-\frac{1}{7}\right]$$

$$=2^{n-1}\left[\frac{21y-9}{28}\right]>0 \; if \; y>0,$$

where the second inequality follows since $n\geq 3$. Thus, we can send $\psi(C_m)-y$ pebbles to v_0 and already C_m contains y pebbles implies that C_m contains $\psi(C_m)$ pebbles and we are done.

So, $\psi(L(m,n)) \leq 2^{n-1}\psi(C_m)+\psi(P_{n-2})$.

Therefore, $\psi(L(m,n)) = 2^{n-1}\psi(C_m)+\psi(P_{n-2})$, if $\alpha_k=2$.

Case3: Let $\alpha_k=0$. Then $k\geq 5$.

Consider the distribution of ψ(L(m,n))-1pebbles at $v_{p_{n-1}}$. Clearly, we cannot cover dominate at least one of the vertices of L(m,n).

Thus, $\psi\,(L(m,n)) \geq 2^{n-1}\psi\,(C_m) + \psi\,(P_{n-2})$.

Now, consider the distribution of ψ(L(m,n)) pebbles on the vertices of L(m,n).

Case3.1 : C_m contains at least $\psi(C_m)$ pebbles.

If P_C contains $\psi(P_{n-1})$ pebbles are more, then clearly we are done(by our assumption). So assume that P_C contains $x< \psi(P_{n-1})$ pebbles. This implies that, C_m contains $2^{n-1}\psi\,(C_m) + \psi\,(P_{n-2}) - x$ pebbles. Suppose, we cannot move $\psi(P_n)$-x pebbles to v_0, then we must have,

$$\hat{f}(P_A) + \hat{f}(P_B) + \left\lfloor \frac{\hat{f}(v_{k-1}) + \left\lfloor \frac{\hat{f}(v_k)}{2} \right\rfloor}{2} \right\rfloor + \left\lfloor \frac{\hat{f}(v_k) + \left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor}{2} \right\rfloor \leq 2^{k-1}\left[\psi\,(P_n) - x\right] - 2. \quad \text{---- (37)}$$

To minimize the L.H.S of (37), it is sufficient to assume that $\hat{f}(P_A)$=0= $\hat{f}(P_B)$. That is, we may assume that all the pebbles are at v_k and v_{k-1}.

Now, $2^{n-1}\psi\,(C_m) + \psi\,(P_{n-2}) - x$ is odd or even, since it depends on both $\psi(P_{n-2})$ and x.

Subcase3 (a) : Suppose, $\hat{f}(v_{k-1}) + \hat{f}(v_k)$ is even.

This implies that, both $\hat{f}(v_{k-1})\, and\, \hat{f}(v_k)$ are odd or even.

Suppose, both $\hat{f}(v_{k-1})\, and\, \hat{f}(v_k)$ are odd, then from (37), we get

$$\frac{3}{4}\left[\hat{f}(v_{k-1}) + \hat{f}(v_k) - 2\right] \le 2^{k-1}\left[\psi(P_n) - x\right] - 2. \qquad \text{---- (38)}$$

But, we have $\hat{f}(v_k) + \hat{f}(v_{k-1}) \ge 2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x$

$$= 2^{n-1}\left[2\psi(P_k) - |\alpha_k - 1|\right] + \psi(P_{n-2}) - x$$

$$\ge 2^{n-1}\left[\frac{2(2^{k+1}-1)}{7} - 1\right] + \frac{2^{n-1}-2}{7} - x,$$

where the second equality follows since m=2k-1, and the third inequality follows since $\alpha_k = 2\, and\, \psi(P_n) \ge \frac{2^{n+1}-2}{7}$.

That is, $\frac{3}{4}\left[\hat{f}(v_k) + \hat{f}(v_{k-1}) - 2\right] \ge \frac{3}{4}\left[2^n\left[\frac{2^{k+1}-1}{7} - \frac{1}{2}\right] + \frac{2^{n-1}-2}{7}\right] - \frac{3}{4}x - \frac{3}{2}$

$$\ge 2^{k-1}\psi(P_n) + 2^{k-2}\psi(P_n) - \frac{3}{4}\left[\frac{3(2^k)}{7} - \frac{8(2^{n-1})}{7}\right] - \frac{3}{4}x - 2$$

$$\ge 2^{k-1}\psi(P_n) + 2^{k-2}\left[\frac{2^{n+1}-2}{7} - \frac{3}{4}\left(\frac{12}{7}\right) - \frac{8(2^{n-1})}{7(8)}\right] - \frac{3}{4}x - 2$$

$$= 2^{k-1}\psi(P_n) + 2^{k-2}\left[\frac{8(2^{n+1}-11) - 8(2^{n-1})}{7}\right] - \frac{3}{4}x - 2$$

$$\geq 2^{k-1}\psi(P_n) + 2^{k-2}\left[\frac{24(2^{n-1}) - 88}{7}\right] - x - 2$$

$$\geq 2^{k-1}\psi(P_n) - x - 2,$$

where the second inequality follows since $\psi(P_n) \leq \frac{2^{n+1}+3}{7}$, the third inequality follows since $k \geq 5\ and\ \psi(P_n) \geq \frac{2^{n+1}-2}{7}$, and the sixth inequality follows $since\ n \geq 3\ and\ k \geq 5.$

That is, $\frac{3}{4}\left[\hat{f}(v_k) + \hat{f}(v_{k-1}) - 2\right] \geq 2^{k-1}\psi(P_n) - x - 2$. ---- (39)

The inequality in (38) contradicts the inequality in (39).

Suppose, both $\hat{f}(v_{k-1})\ and\ \hat{f}(v_k)$ are even, then from (37), we get

$$\frac{3}{4}\left[\hat{f}(v_{k-1}) + \hat{f}(v_k)\right] \leq 2^{k-1}\left[\psi(P_n) - x\right] - 1. \qquad \text{---- (40)}$$

But, we have $\hat{f}(v_k) + \hat{f}(v_{k-1}) \geq 2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x$

$$= 2^{n-1}\left[2\psi(P_k) - |\alpha_k - 1|\right] + \psi(P_{n-2}) - x$$

$$\geq 2^{n-1}\left[\frac{2(2^{k+1}-1)}{7} - 1\right] + \frac{2^{n-1}-2}{7} - x$$

$\geq 2^k\psi(P_n) - \frac{3(2^k)}{7} - \frac{8(2^{n-1})}{7} - \frac{2}{7} - x,$ where the second equality follows since m=2k-1, the third inequality follows since $\alpha_k = 2\ and\ \psi(P_n) \geq \frac{2^{n+1}-2}{7}$.

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right] \geq \frac{3}{4}\left[2^k\psi(P_n)-\frac{3(2^k)}{7}-\frac{8(2^{n-1})}{7}-\frac{2}{7}-x\right]$

$\geq 2^{k-1}\psi(P_n)+2^{k-2}\left[\frac{24(2^{n-1})-88}{7}\right]-x-2$

$\geq 2^{k-1}\psi(P_n)-x-1,$

where the third inequality follows $since\ n \geq 3\ and\ k \geq 5.$

That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right] \geq 2^{k-1}\psi(P_n)-x-1.$ ---- (41)

The inequality in (40) contradicts the inequality in (41).

Subcase3 (b) : If $\hat{f}(v_{k-1})+\hat{f}(v_k)$ is odd.

Without loss of generality, let $\hat{f}(v_k)$ be odd. Then $\hat{f}(v_{k-1})$ is even.

From (37), we get

$\frac{3}{4}\left[\hat{f}(v_{k-1})+\hat{f}(v_k)\right]-\frac{5}{4} \leq 2^{k-1}\left[\psi(P_n)-x\right]-2.$ ---- (42)

But, we have $\hat{f}(v_k)+\hat{f}(v_{k-1}) \geq 2^{n-1}\psi(C_m)+\psi(P_{n-2})-x$

$=2^{n-1}\left[2\psi(P_k)-|\alpha_k-1|\right]+\psi(P_{n-2})-x$

$\geq 2^k\psi(P_n)-\frac{3(2^k)}{7}-\frac{8(2^{n-1})}{7}-\frac{2}{7}-x,$

where the second equality follows since m=2k-1. That is, $\frac{3}{4}\left[\hat{f}(v_k)+\hat{f}(v_{k-1})\right]-\frac{5}{4} \geq \frac{3}{4}\left[2^k\psi(P_n)-\frac{3(2^k)}{7}-\frac{8(2^{n-1})}{7}-\frac{2}{7}-x\right]-\frac{3}{4}x-\frac{5}{4}$

$$\geq 2^{k-1}\psi(P_n) + 2^{k-2}\left[\frac{24(2^{n-1}) - 88}{7}\right] - x - 2$$

$$\geq 2^{k-1}\psi(P_n) - x - 2,$$

where the third inequality follows since n>2 and k>4.

That is, $\frac{3}{4}\left[\hat{f}(v_k) + \hat{f}(v_{k-1})\right] - \frac{5}{4} \geq 2^{k-1}\psi(P_n) - x - 2$. ---- (43)

The inequality in (42) contradicts the inequality in (43).

From Subcase3 (a) and Subcase3 (b), we can always send $\psi(P_n) - x$ pebbles to v_0 at a cost of at most $2^{k-1}[\psi(P_n) - x]$ pebbles. Thus, we cover dominate the path P_n. Now, we have to cover dominate C_m. In C_m, we have at least $2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}\left[\psi(P_n) - x\right]$ pebbles. We need at most $\psi(C_m)$ pebbles to cover dominate C_m. But,

$$2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}\left[\psi(P_n) - x\right] - \psi(C_m)$$

$$= (2^{n-1} - 1)\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}\left[\psi(P_n) - x\right]$$

$$= (2^{n-1} - 1)\left[2\psi(P_k) - 1\right] - 2^{k-1}\psi(P_n) + \psi(P_{n-2}) + (2^{k-1} - 1)x$$

$$\geq (2^{n-1} - 1)\left(\frac{2(2^{k+1} - 1)}{7} - 1\right) - 2^{k-1}\left(\frac{2^{n+1} + 3}{7}\right)$$

$$\geq 2^{k-1}\left[\frac{2(2^n) - 11}{7} - \frac{9(2^{n-1} - 1)}{7(16)}\right]$$

$$\geq 2^{k-1}\left[\frac{55(2^{n-1}) - 165}{56}\right] > 0,$$

where the second equality follows since m=2k-1, the third inequality follows since $\alpha_k = 0 \; and \, \psi(P_n) \le \frac{2^{n+1}+3}{7}$, the fifth inequality follows since k≥5, and the sixth inequality follows $since \; n > 2 \, and \; k > 4$.

Thus, we have enough pebbles to cover dominate C_m and hence we are done.

Case3.2: C_m contains $y<\psi(C_m)$ pebbles.

This implies that, P_C contains $2^{n-1}\psi(C_m)+\psi(P_{n-2})-y$ pebbles. We use at most $\psi(P_{n-1})$ pebbles to cover dominate P_C. Thus, we have at least $2^{n-1}\psi(C_m)+\psi(P_{n-2})-y-\psi(P_{n-1})$ pebbles in P_C. We need at most $2^{n-1}[\psi(C_m)-y]$ pebbles from P_C to cover dominate the vertices of C_m. But,

$$2^{n-1}\psi(C_m)+\psi(P_{n-2})-y-\psi(P_{n-1})-2^{n-1}\left[\psi(C_m)-y\right]$$

$$=2^{n-1}y+\psi(P_{n-2})-y-\psi(P_{n-1})$$

$$\ge 2^{n-1}\left[y-\frac{5+7y}{7(4)}-\frac{1}{7}\right]$$

$$=2^{n-1}\left[\frac{21y-9}{28}\right]>0 \; if \;\; y>0,$$

where the third inequality follows since n≥3. Thus, we can send $\psi(C_m)$-y pebbles to v_0 and already C_m contains y pebbles implies that C_m contains $\psi(C_m)$ pebbles and we are done.

So, $\psi(L(m,n)) \le 2^{n-1}\psi(C_m)+\psi(P_{n-2})$.

Therefore, $\psi(L(m,n)) = 2^{n-1}\psi(C_m)+\psi(P_{n-2})$, if α_k=0.

Hence, $\psi(L(m,n)) = \begin{cases} 2^{n-1}\psi(C_m) + \psi(P_{n-1}), if\ \alpha_k = 1 \\ 2^{n-1}\psi(C_m) + \psi(P_{n-2}), if\ \alpha_k = 0\ or 2 \end{cases}$, where m=2k-1 and k-2 $\equiv \alpha_k$(mod 3). ■

References

[1] F.R.K. Chung, Pebbling in Hypercubes, SIAM J. Discrete Mathematics 2 (1989), 467-472.

[2] B. Crull, T. Cundiff, P. Feltman, G. Hulbert, L.Pudwell, Z.Szaniszlo, and T.Suza, The Cover Pebbling Number of Graphs, Discrete Math.296 (2005), 15-23.

[3] J. Gardner, A. Godbole, A. Teguia, A. Vuong, N. Watson and C. Yerger, Domination Cover Pebbling: Graph Families, JCMCC 64 (2008), 255-271.

[4] T. Haynes, S. Hedetniemi, and P. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[5] G. Hulbert, A Survey of Graph Pebbling, Congr. Numer.139 (1999), 41-64.

[6] KM. Kathiresan and G. Marimuthu, Superior Domination in Graphs, Utilitas Mathematica 76 (2008), 173-182.

[7] A. Lourdusamy and T. Mathivanan,The Domination Cover Pebbling Number of the Square of a Path, Journal of Prime Research in Mathematics, Vol. 7, Nov. 2011, 1-8.